JOURNAL OF COMPUTATIONAL PHYSICS 93, 477–480 (1991)

Note

A Riemann Solver for "Barotropic" Flow

1. Introduction

In 1988 Glaister [1] proposed an approximate linearised Riemann solver for the Euler equations of gas dynamics in one dimension. In certain applications, e.g., flow of natural gas in a pipe, it is not necessary to use the full Euler equations, but to replace the energy equation by the algebraic statement of a one-to-one relationship between pressure and density. Some authors have used the term "barotropic" to describe such flows (see, e.g., [2]). We note here the corresponding scheme to [1] for barotropic flow.

2. Equations of Flow

2.1. *Equations of Motion*

The one-dimensional equations of barotropic flow can be written in conservation form as

$$\mathbf{w}_t + \mathbf{f}_x = \mathbf{0}, \tag{2.1}$$

where

$$\mathbf{w} = (\rho, \rho u)^{\mathrm{T}} \tag{2.2}$$

and

$$\mathbf{f} = (\rho u, p + \rho u^2)^{\mathrm{T}}. \tag{2.3}$$

The quantities $(\rho, u, p) = (\rho, u, p)(x, t)$ represent the density, velocity, and pressure at a general position x in space and at time t. In addition, we assume that there is a convex gas law relating p and ρ written as

$$p = p(\rho). \tag{2.4}$$

We assume further that the derivative $dp/d\rho$ of the gas law (2.4) can be determined. For a polytropic gas $p = \text{const} \times \rho^n$, n const. In particular, $n = 1$ gives isothermal flow, $n = \gamma =$ ratio of specific heat capacities of the fluid gives isentropic flow, and $n = 2$ gives equations analogous to the shallow water equations. Another interesting

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application is that of compressible flow of water, for which the equation of state is sometimes approximated as

$$p/p_0 = 3001(\rho/\rho_0)^7 - 3000,$$

where $(\ )_0$ denote STP (standard temperature and pressure) conditions (see Courant and Friedrichs [3]).

2.2. *Jacobian*

The Jacobian matrix $A = \partial \mathbf{f}/\partial \mathbf{w}$ has eigenvalues

$$\lambda_j = u \pm a, \qquad j = 1, 2 \tag{2.5a)–(2.5b}$$

with corresponding eigenvectors

$$\mathbf{e}_{1,2} = (1, u \pm a)^{\mathrm{T}}, \tag{2.6a)–(2.6b}$$

where the sound speed a is given by

$$a^2 = \frac{dp}{d\rho}, \tag{2.7}$$

using Eq. (2.4).

3. Approximate Riemann Solver

In this section we state an approximate Riemann solver for the barotropic flow in one dimension with a general convex gas law. We make a brief comparison with the scheme of Glaister [1] for the Euler equations.

3.1. *Wavespeeds for Nearby States*

As in [1], we consider the solution at any time to consist of a series of piecewise constant states. Our aim is then to solve each of these linearised Riemann problems approximately. Consider two (constant) adjacent states $\mathbf{w}_L$, $\mathbf{w}_R$ (left and right) close to an average state $\mathbf{w}$, at points L and R on an x-coordinate line. We assume that we have approximate eigenvectors

$$\mathbf{r}_{1,2} = (1, u \pm a)^{\mathrm{T}} \tag{3.1a)–(3.1b}$$

corresponding to the average state $\mathbf{w}$.

As usual we seek coefficients α_1, α_2 such that

$$\Delta \mathbf{w} = \alpha_1 \mathbf{r}_1 + \alpha_2 \mathbf{r}_2 \tag{3.2a)–(3.2b}$$

to within $O(\Delta^2)$, where $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$. Solving Eq. (3.2a)–(3.2b) we obtain

$$\alpha_{1,2} = \frac{1}{2a^2}\left(a^2\, \Delta\rho \pm a(\Delta(\rho u) - u\, \Delta\rho)\right), \tag{3.3a)–(3.3b}$$

whereas the corresponding expressions in [1] are

$$\alpha_{1,2}=\frac{1}{2a^2}\,(\Delta p \pm \rho a\,\Delta u). \tag{3.4a)–(3.4b}$$

3.2. *Decomposition for General* $\mathbf{w}_L$, $\mathbf{w}_R$

The Riemann solver is now constructed by finding average eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2$ and corresponding average eigenvectors $\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2$ such that

$$\Delta\mathbf{w}=\sum_{j=1}^{2} \tilde{\alpha}_j \tilde{\mathbf{r}}_j \tag{3.5}$$

and

$$\Delta\mathbf{f}=\sum_{j=1}^{2} \tilde{\lambda}_j \tilde{\alpha}_j \tilde{\mathbf{r}}_j. \tag{3.6}$$

where

$$\tilde{\lambda}_{1,2}=\tilde{u}\pm\tilde{a}, \tag{3.7a)–(3.7b}$$

$$\tilde{\mathbf{r}}_{1,2}=(1,\,\tilde{u}\pm\tilde{a})^{\mathrm{T}}, \tag{3.8a)–(3.8b}$$

and

$$\tilde{\alpha}_{1,2}=\frac{1}{2\tilde{a}^2}\,(\tilde{a}^2\,\Delta\rho \pm \tilde{a}(\Delta(\rho u)-\tilde{u}\,\Delta\rho)). \tag{3.9a)–(3.9b}$$

Thus, we have to determine averages $\tilde{\rho}$, $\tilde{u}$, and $\tilde{a}$. This type of construction was originally considered by Roe and Pike [4] for the Euler equations with ideal gases and subsequently used by Glaister [1] for the Euler equations with more general equations of state.

Solving, we find

$$\tilde{u}=\frac{\sqrt{\rho_L}\,u_L+\sqrt{\rho_R}\,u_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}, \tag{3.10}$$

$$\begin{aligned} \tilde{a}^2&=\frac{\Delta p}{\Delta\rho}=\frac{p(\rho_R)-p(\rho_L)}{\rho_R-\rho_L}, &\quad \Delta\rho\neq 0,\quad \rho_R\neq\rho_L \\ \tilde{a}^2&=\frac{dp}{d\rho}\,(\rho) &\quad \Delta\rho=0,\quad \rho_R=\rho_L. \end{aligned} \tag{3.11}$$

This is similar to [1], where $\tilde{a}^2$ is related to the derivatives to the equation of state. (N.B. In practice $\Delta\rho=0$ is replaced by $|\Delta\rho|\leqslant 10^{-m}$, where the integer m is machine dependent.) In addition, from Eq. (3.10) we can show that

$$\Delta(\rho u)-\tilde{u}\,\Delta\rho=\sqrt{\rho_L\rho_R}\,\Delta u, \tag{3.12}$$

so that defining the average

$$\tilde{\rho} = \sqrt{\rho_L \rho_R} \tag{3.13}$$

simplifies the expressions in (3.9a)–(3.9b) to

$$\tilde{\alpha}_{1,2} = \frac{1}{2\tilde{a}^2} (\tilde{a}^2 \, \Delta\rho \pm \tilde{\rho}\tilde{a} \, \Delta u). \qquad \text{(3.14a)–(3.14b)}$$

4. Conclusions

We have given a Riemann solver similar to that of Glaister [1] that applies to "barotropic" flow with a general convex gas law. As usual we retain the important shock capturing property prevalent in this type of scheme. In particular, the scheme can be applied to barotropic flows where it is not necessary to use the full Euler equations.

References

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Received: March 10, 1989; revised: January 16, 1990

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